

The constant objective value property for combinatorial optimization problems

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Abstract

Given a combinatorial optimization problem, we aim at characterizing the set of all instances for which every feasible solution has the same objective value.

Our central result deals with multi-dimensional assignment problems. We show that for the axial and for the planar d -dimensional assignment problem instances with constant objective value property are characterized by sum-decomposable arrays. We provide a counterexample to show that the result does not carry over to general d -dimensional assignment problems.

Our result for the axial d -dimensional assignment problem can be shown to carry over to the axial d -dimensional transportation problem. Moreover, we obtain characterizations when the constant objective value property holds for the minimum spanning tree problem, the shortest path problem and the minimum weight maximum cardinality matching problem.

Keywords. Constant objective value; admissible transformation; multi-dimensional assignment problem; sum-decomposable array.

1 Introduction

In this paper we deal with combinatorial optimization problems of the following type. We are given a ground set $E = \{1, \dots, n\}$, a real cost vector $C = (c(1), \dots, c(n))$ and a set of feasible solutions $\mathcal{F} \subseteq 2^{\{1, \dots, n\}}$. The *objective value* of a feasible solution $F \in \mathcal{F}$ is given by the so-called *sum objective function*

$$c(F) := \sum_{i \in F} c(i).$$

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The goal is to find a feasible solution F^* such that $c(F^*)$ is minimal. The traveling salesman problem, the linear assignment problem, the shortest path problem, Lawler's quadratic assignment problem and many other well-known combinatorial optimization problems fall into this class of problems.

Definition 1.1. We say that an instance of a combinatorial optimization problem has the *constant objective value property* (COVP) if every feasible solution has the same objective value.

Our goal is to characterize the set of instances with the COVP, or in other words, the space of all cost vectors for which every feasible solution has the same objective value, for various combinatorial optimization problems.

Related results. The constant objective value property is closely connected to the notion of *admissible transformations* introduced in 1971 by Vo-Khac [20].

Definition 1.2. A transformation T of the cost vector C to the new cost vector $\tilde{C} = (\tilde{c}(1), \tilde{c}(2), \dots, \tilde{c}(n))$ is called admissible with index $z(T)$, if

$$c(F) = \tilde{c}(F) + z(T) \quad \text{for all } F \in \mathcal{F}.$$

Note that admissible transformations preserve the relative order of the objective values of all feasible solutions. It is well known that admissible transformations can be used as optimality criterion and to obtain lower bounds which are useful for hard combinatorial optimization problems. Namely, consider the combinatorial optimization problem $\min_{F \in \mathcal{F}} c(F)$. Let T be an admissible transformation with index $z(T)$ from the original cost vector C to the new cost vector \tilde{C} such that there exists a feasible solution F^* with the following properties:

- (i) $\tilde{c}(i) \geq 0$ for all $i \in \{1, \dots, n\}$,
- (ii) $\tilde{c}(F^*) = 0$.

Then F^* is an optimal solution with objective value $z(T)$. If the condition (ii) is not satisfied or we cannot prove that it holds, then $z(T)$ gives a lower bound.

For the class of combinatorial optimization problems with sum objective function there is a one-to-one correspondence between admissible transformations that transform the cost vector $(c(1), \dots, c(n))$ to $(\tilde{c}(1), \dots, \tilde{c}(n))$, and cost vectors $B = (b(1), b(2), \dots, b(n))$ that fulfill the COVP. The correspondence is obtained by $c(i) = \tilde{c}(i) + b(i)$ for all i . Then the index of the corresponding admissible transformation is $z(T) = \sum_{i \in F} b(i)$ for any $F \in \mathcal{F}$. The correspondence between the COVP and admissible transformations provides a further source of motivation for investigating COVP characterizations.

The notion of admissible transformations can be generalized to the algebraic setting and applied to a wider class of combinatorial optimization problems, including the case of bottleneck objective functions, see [6]. Note, however, that for the bottleneck objective function, which is given by $c(F) = \max_{i \in F} c(i)$, there is no one-to-one correspondence between the COVP and admissible transformations.

Berenguer [2] characterized the set of all admissible transformations for the travelling salesman problem (TSP) and the multiple salesmen version. All admissible transformations for the TSP are obtained by adding real values to rows and columns of the distance matrix. In view of the correspondence mentioned above this result can be rephrased as a result on the COVP for the TSP as follows (this has been noted already by Gilmore, Lawler and Shmoys [11]).

An $n \times n$ real matrix $C = (c_{ij})$ is called *sum matrix* if there exist two real n -dimensional vectors $U = (u_i)$ and $V = (v_i)$ such that

$$c_{ij} = u_i + v_j \quad \text{for all } i, j \in \{1, \dots, n\}. \quad (1)$$

Theorem 1.1 (Berenguer [2], Gilmore et al. [11]). *The TSP instance with the $n \times n$ cost matrix $C = (c_{ij})$ has the COVP if and only if C is a sum matrix.*

For the TSP the diagonal entries of C do not play a role and can be ignored. Berenguer's proof works for the linear assignment problem as well, i.e. an instance of the linear assignment problem with cost matrix $C = (c_{ij})$ has the COVP if and only if C is a sum matrix.

Some classes of admissible transformations for different types of assignment problems are listed by Burkard [4]. However, no COVP characterizations are provided.

We remark that there is a simpler way to prove the COVP characterization for the linear assignment problem mentioned above by making use of the LP-duality and the complementary slackness condition. Since for each pair (i, j) there exists an assignment which assigns i to j (i.e. the primal assignment variable x_{ij} is 1), all dual constraints need to be fulfilled with equality which is equivalent to the condition (1) (note that the vectors U and V contain the dual variables).

Results and organization of the paper. In Section 2 we investigate the problem of characterizing the instances with the COVP for multi-dimensional assignment problems. We show that for the multi-dimensional axial and planar case the cost arrays with the COVP are precisely the class of sum-decomposable arrays which are generalizations of sum matrices (for the precise definition see Section 2). We furthermore provide a counterexample which shows that sum-decomposability is not necessarily required for the COVP to hold for general multi-dimensional assignment problems.

In Section 3 the result for the axial d -dimensional assignment problem is carried over to the axial d -dimensional transportation problem. Finally, in Section 4 we deal with COVP characterizations for the minimum spanning tree problem, the shortest path problem and the minimum weight maximum cardinality matching problem.

2 The COVP for d -dimensional assignment problems

Berenguer's result for the classical linear assignment problem motivated us to ask for COVP characterizations for multi-dimensional assignment problems.

2.1 General multi-dimensional assignment problems

Two classical ways of generalizing the notion of assignments to three dimensions are the so-called *axial 3-dimensional (or 3-index) assignment problem* and the *planar 3-dimensional (or 3-index) assignment problem*. A further generalization is obtained by the class of d -dimensional assignment problems defined in the sequel. For more on the topic of assignment problems see the book by Burkard et al. [5] and the references cited therein.

A general multi-dimensional assignment problem is specified by two parameters d and s , where d is the number of indices and s describes the number of fixed indices in the constraints. Informally speaking, we want to find a set of n^s elements of a d -dimensional $n \times n \times \cdots \times n$ array C with minimal total sum, such that for every s fixed indices of C exactly one element is chosen.

Formally, the (d, s) assignment problem, (d, s) -AP for short, can be stated in the following way.

Definition 2.1. Let d and s be integers with $0 < s < d$. The input of the (d, s) -AP consists of an integer $n \geq 1$ and a d -dimensional $n \times n \times \cdots \times n$ cost array C which associates the cost $c(i_1, i_2, \dots, i_d)$ to the d -tuple $(i_1, i_2, \dots, i_d) \in \{1, \dots, n\}^d$. Let \mathcal{Q}_s be the set of all subsets of $K = \{1, \dots, d\}$ with cardinality s , i.e. $\mathcal{Q}_s = \{Q: Q \subset K, |Q| = s\}$. For any set $Q = \{q_1, q_2, \dots, q_s\} \in \mathcal{Q}_s$ of fixed indices with $q_1 < q_2 < \cdots < q_s$ and any s -tuple $t = (t_1, \dots, t_s) \in \{1, \dots, n\}^s$, let $T(Q, t)$ be the set of all d -tuples $t' = (t'_1, \dots, t'_d) \in \{1, \dots, n\}^d$ such that $t'_{q_j} = t_j$ for all $j = 1, \dots, s$.

The general (d, s) assignment problem (d, s) -AP can be stated as

$$\begin{aligned}
& \min \sum_{i_1=1}^n \cdots \sum_{i_d=1}^n c(i_1, i_2, \dots, i_d) x(i_1, i_2, \dots, i_d) \\
& \text{s.t.} \quad \sum_{\substack{(i_1, \dots, i_d) \in \\ T(Q, (j_1, \dots, j_s))}} x(i_1, i_2, \dots, i_d) = 1 \text{ for all } Q \in \mathcal{Q}_s \\
& \quad \text{and all } (j_1, \dots, j_s) \in \{1, \dots, n\}^s, \\
& \quad x(i_1, i_2, \dots, i_d) \in \{0, 1\} \text{ for all } (i_1, i_2, \dots, i_d) \in \{1, \dots, n\}^d.
\end{aligned} \tag{2}$$

Note that in each of the equality constraints above, the sum essentially extends over $d - s$ variables (corresponding to the free indices from the set $K \setminus Q$).

Let $X = (x(i_1, i_2, \dots, i_d))$ be a feasible solution of the integer program stated above. Then the set $F = \{(i_1, i_2, \dots, i_d) : x(i_1, i_2, \dots, i_d) = 1\}$ is a feasible solution of the (d, s) -AP and the value $\sum_{(i_1, \dots, i_d) \in F} c(i_1, \dots, i_d)$ is the cost (objective value) of a feasible solution F . Hence the (d, s) -AP fits into the class of combinatorial optimization problems with sum objective function.

Using the (d, s) -AP notation, the classical linear assignment problem is the $(2, 1)$ -AP, while the axial and the planar 3-dimensional assignment problems correspond to the $(3, 1)$ -AP and the $(3, 2)$ -AP, respectively. More generally, we refer to the $(d, 1)$ -AP as *axial d -dimensional assignment problem*, and to the $(d, d - 1)$ -AP as *planar d -dimensional assignment problem*. Let us remark that for $d \geq 4$, there is no consensus in the literature which problem version is referred to as planar d -dimensional assignment problem. Our decision to refer to the $(d, d - 1)$ -AP in which all constraints involve single sums as to planar is in accordance with the axial/planar nomenclature which was introduced in the original paper by Schell [18] and is used by Spieksma [19], Frieze and Sorkin [9] and others. A group of authors around Appa, see e.g. [1], refer to the $(d, 2)$ -AP as to planar d -dimensional assignment problem. The axial and the planar 3-dimensional assignment problems are known to be NP-hard [10, 12]. As a consequence thereof both the axial and the planar d -dimensional assignment problems are NP-hard for all $d \geq 3$.

The following observations collect a few well known facts about the structure of the set of feasible solutions of the (d, s) -AP. These will turn out to be helpful in later parts of the paper.

Observation 2.1. Every feasible solution of the $(d, 1)$ -AP of size n can be represented by a set of $d - 1$ permutations of $\{1, \dots, n\}$. Namely, every feasible solution can be written as $F = \{(i, \phi_1(i), \dots, \phi_{d-1}(i)) : i = 1, \dots, n\}$ where each ϕ_k is a permutation of $\{1, \dots, n\}$.

Observation 2.2. A set of d -tuples F is a feasible solution of the $(d, d-1)$ -AP of size n if and only if F “contains” n pairwise disjoint feasible solutions of the $(d-1, d-2)$ -AP. Namely, define n sets of $(d-1)$ -tuples obtained from F by fixing one index, for example the first one: $F_i = \{(a_2, a_3, \dots, a_d) : (i, a_2, a_3, \dots, a_d) \in F\}$ for $i = 1, 2, \dots, n$. Then every F_i is a feasible solution of the $(d-1, d-2)$ -AP because fixing $d-2$ indices in F_i corresponds to fixing $d-1$ indices in F . Also, if there are F_i and F_j that are not disjoint, then there would be two elements in F that coincide on $d-1$ indices, a contradiction. The same construction works in the other direction as well.

Observation 2.3. The feasible solutions of the $(d, 2)$ -AP for $d \geq 3$ correspond to $(d-2)$ -tuples of mutually orthogonal Latin squares. Indeed, assume that F is a feasible solution of the $(d, 2)$ -AP. We can represent F as an $n \times n$ table T with $(d-2)$ -tuples as entries in the following way: The $(d-2)$ -tuple (i_3, \dots, i_d) is the entry in row i_1 and column i_2 of T if and only if (i_1, \dots, i_d) is an element of F . Since F is a feasible solution of the $(d, 2)$ -AP, each row and each column of T contain every integer from 1 to n exactly once on the k -position for all $k = 1, \dots, d-2$. Moreover, each $(d-2)$ -tuple of pairwise distinct integers from $\{1, \dots, n\}$ appears exactly once in T . Hence T can be interpreted as a $(d-2)$ -tuple of mutually orthogonal Latin squares (the k -th component of the entries of T yields the k -th Latin square). Note that if $d = 4$ then T is a Graeco-Latin square.

What distinguishes the general (d, s) -AP from the special cases with $s = 1$ (axial problem) and with $s = d-1$ (planar problem) is that there does not need to exist feasible solutions for every value of n . Furthermore, not much is known on the structure of the set of feasible solutions for the (d, s) -AP for general n , cf. Appa et al. [1]. Infeasible instances and instances with very few feasible solutions provide clear obstacles to our intended COVP characterization. For this reason the feasibility topic for the (d, s) -AP plays a role for us and we briefly review a few basic results from the literature.

The question for which values n the general (d, s) -AP has feasible solutions is a very difficult problem which is still open for many combinations of d and s , and is related to a number of difficult problems in combinatorics. More precisely, the general (d, s) -AP of size n has a feasible solution if there exists an s -transversal design with d groups of size n , or equivalently if there exists an orthogonal array $\text{OA}(n, d, s)$ of index 1, strength s and order n , see [7].

In view of Observation 2.3, the question of existence of Graeco-Latin squares is of interest. This was a famous open problem for a long time going back to Euler until it was finally proved by Bose et al. [3] that Graeco-Latin squares, and thus feasible solutions of the $(4, 2)$ -AP, exist for every $n \geq 3$ except for $n = 6$.

The number of mutually orthogonal Latin squares of size n (see Observation 2.3 for the connection to the feasibility of the $(d, 2)$ -AP) is unknown for general n . It is known however that this number is at most $n - 1$ and that the upper bound is achieved if n is a prime power. It is also known that there exist $n - 1$ mutually orthogonal Latin squares if and only if there exists a projective plane of order n , see [7].

2.2 Sum-decomposable arrays

In this subsection we investigate the vector spaces of *sum-decomposable arrays*. These will occur as solutions of various COVP characterizations. Sum-decomposable arrays generalize the concept of sum matrices to higher dimensions.

Informally, a d -dimensional $n \times n \times \cdots \times n$ real array C is sum-decomposable with parameters d and s (and size n) if C can be obtained as a sum of $\binom{d}{s}$ s -dimensional arrays, one for each subset of $\{1, \dots, d\}$ of size s . For example, in the case $d = 3$ and $s = 2$, $C = (c_{ijk})$ is sum-decomposable if there exist three two-dimensional real arrays $A = (a_{ij})$, $B = (b_{ij})$ and $D = (d_{ij})$ such that $c_{ijk} = a_{ij} + b_{ik} + d_{jk}$. A formal definition follows.

Definition 2.2. Let n , d and s be integers such that $d > s > 0$ and $n > 1$. Let $\mathcal{Q}_s = \{Q: Q \subset \{1, \dots, d\}, |Q| = s\}$. Then the d -dimensional $n \times n \times \cdots \times n$ real array C is called *sum-decomposable with parameters d and s and size n* if there exist $\binom{d}{s}$ s -dimensional $n \times n \times \cdots \times n$ real arrays $A^Q = (a^Q(j_1, \dots, j_s))$, one for each $Q \in \mathcal{Q}_s$, such that

$$c(i_1, i_2, \dots, i_d) = \sum_{Q \in \mathcal{Q}_s} a^Q(h_Q(i_1, i_2, \dots, i_d))$$

where $h_Q(i_1, i_2, \dots, i_d)$ denotes the s -tuple associated with Q , i.e. $h_Q(i_1, i_2, \dots, i_d) = (i_{q_1}, \dots, i_{q_s})$ for $Q = \{q_1, \dots, q_s\}$, $q_1 < q_2 < \cdots < q_s$.

We denote the vector space of all sum-decomposable real arrays of size n with parameters d and s by $\text{SAVS}(d, s, n)$.

For $Q = \{j_1, j_2, \dots, j_s\} \in \mathcal{Q}_s$ let V_Q denote the vector space of all d -dimensional $n \times n \times \cdots \times n$ arrays $C = (c(i_1, i_2, \dots, i_d))$ for which there exists a mapping $f: \{1, 2, \dots, n\}^s \mapsto \mathbb{R}$ with $c(i_1, i_2, \dots, i_d) = f(i_{j_1}, i_{j_2}, \dots, i_{j_s})$. In other words, the value $c(i_1, i_2, \dots, i_d)$ depends only on the s indices from the set Q and not on all d indices. Let $Q_1, Q_2, \dots, Q_{\binom{d}{s}}$ be such that $\mathcal{Q}_s = \{Q_1, \dots, Q_{\binom{d}{s}}\}$. Note that

$$\text{SAVS}(d, s, n) = V_{Q_1} + V_{Q_2} + \cdots + V_{Q_{\binom{d}{s}}}. \quad (3)$$

We will use the following proposition, that can be found in [15, Prop. 7.1 of Chap. 1, p. 15], to prove that a variant of inclusion-exclusion principle

holds for the vector subspaces V_{Q_i} . Recall that the distributivity with respect to sum and intersection does not hold for arbitrarily vector subspaces, i.e. it is not true that $V_1 \cap (V_2 + V_3) = (V_1 \cap V_2) + (V_1 \cap V_3)$ holds for all vector spaces V_1, V_2, V_3 .

Proposition 2.4. *Let W be a vector space and $V_1, V_2, \dots, V_n \subset W$ be a collection of its subspaces. Then the following conditions are equivalent:*

- (i) *The collection V_1, V_2, \dots, V_n is distributive with respect to the operations of sum and intersection.*
- (ii) *There exists a basis $\{w_\alpha : \alpha \in A\}$ of the vector space W such that each of the subspaces V_i is the linear span of a set of vectors w_α .*

Now we are ready to prove the following proposition.

Proposition 2.5. *Let $\text{SAVS}(d, s, n)$ be expressed as in (3). Then we have*

- (i) $\dim(V_Q) = n^s$ for all $Q \in \mathcal{Q}_s$,
- (ii) $\dim(\cap_{i \in I} V_{Q_i}) = n^{|\cap_{i \in I} Q_i|}$ for all $I \subset \{1, \dots, \binom{d}{s}\}$,
- (iii) $\dim(\text{SAVS}(d, s, n)) = \dim\left(\sum_{i=1}^{\binom{d}{s}} V_{Q_i}\right) =$

$$\sum_{k=1}^{\binom{d}{s}} (-1)^{k+1} \left(\sum_{1 \leq i_1 < \dots < i_k \leq \binom{d}{s}} \dim(V_{Q_{i_1}} \cap \dots \cap V_{Q_{i_k}}) \right),$$
- (iv) $\dim(\text{SAVS}(d, d-1, n)) = n^d - (n-1)^d$,
 $\dim(\text{SAVS}(d, 1, n)) = dn - d + 1.$

Proof. Ad (i): Follows directly from the definition of V_Q .

Ad (ii): Let us start with $|I| = 2$ and consider $J = \{j_1, j_2, \dots, j_s\}$, $K = \{k_1, k_2, \dots, k_s\}$ from \mathcal{Q}_s . Further, let C and E be two arrays from V_J and V_K , respectively. Hence, $c(i_1, i_2, \dots, i_d) = f(i_{j_1}, i_{j_2}, \dots, i_{j_s})$ and $e(i_1, i_2, \dots, i_d) = g(i_{k_1}, i_{k_2}, \dots, i_{k_s})$ for some $f, g: \{1, 2, \dots, n\}^s \mapsto \mathbb{R}$. Then for every $A = (a(i_1, \dots, i_d)) \in V_J \cap V_K$ we have that $a(i_1, i_2, \dots, i_d) = t(i_{q_1}, \dots, i_{q_{|J \cap K|}})$, $q_i \in J \cap K$, for some $t: \{1, 2, \dots, n\}^{|J \cap K|} \mapsto \mathbb{R}$. Hence, $\dim(V_J \cap V_K) = n^{|J \cap K|}$. The case $|I| \geq 3$ follows by an inductive argument. This settles (ii).

Ad (iii): The case of two vector spaces involved in the sum follows from the fact that

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2) \quad (4)$$

holds for any two subspaces V_1 and V_2 of a vector space. Next we settle the case of three vector spaces, so let $Q_i, Q_j, Q_\ell \in \mathcal{Q}_s$. The general case then follows by induction. Note that (4) implies that

$$\begin{aligned} \dim(V_{Q_i} + V_{Q_j} + V_{Q_\ell}) = \\ \dim(V_{Q_i}) + \dim(V_{Q_j} + V_{Q_\ell}) - \dim(V_{Q_i} \cap (V_{Q_j} + V_{Q_\ell})). \end{aligned} \quad (5)$$

To proceed further it suffices to prove that the subspaces V_{Q_i} are distributive with respect to sum and intersection, i.e. that

$$V_{Q_i} \cap (V_{Q_j} + V_{Q_\ell}) = (V_{Q_i} \cap V_{Q_j}) + (V_{Q_i} \cap V_{Q_\ell}) \quad (6)$$

holds. It is easy to check that using (6) and (4) in (5) above leads to the claim (iii) for three vector spaces. So, it only remains to show that the distributivity property (6) holds. To that end, we construct bases for the vector spaces V_Q for $Q \in \mathcal{Q}_s$ and for the vector space V of all d -dimensional $n \times \dots \times n$ real arrays. The distributivity then follows from Proposition 2.4. Namely, call an array from V *elementary* if a single entry is 1 and all other entries are 0. It is easy to see that the set of n^d d -dimensional $n \times n \times \dots \times n$ elementary arrays forms a basis for V . Next we construct a basis for the subspace V_Q where $Q = \{j_1, j_2, \dots, j_s\}$. Let A_{k_1, \dots, k_s} be the 0-1 d -dimensional array such that the entry at position (i_1, \dots, i_d) is 1 if $i_{j_1} = k_1, \dots, i_{j_s} = k_s$ and 0 otherwise. Then the set of arrays $\{A_{k_1, \dots, k_s} : (k_1, \dots, k_s) \in \{1, \dots, n\}^s\}$ forms a basis for V_Q . Note that every element of the basis for V_Q can be written as a linear combination of elementary arrays.

Ad (iv): Note that any intersection of ℓ distinct subsets of $\{1, 2, \dots, d\}$ has cardinality $d - \ell$. Hence, from (ii) and (iii) it follows that

$$\dim(\text{SAVS}(d, d-1, n)) = \sum_{i=1}^d (-1)^{i+1} \binom{d}{i} n^{d-i},$$

which is equal to $n^d - (n-1)^d$ by the binomial theorem. Since the intersection of any distinct one-element sets is empty it follows that $\dim(\text{SAVS}(d, 1, n)) = dn - d + 1$. \square

2.3 The COVP for the axial case: $(d, 1)$ -AP

Now we turn to the problem of characterizing the instances of the axial d -dimensional assignment problem with the constant objective value property (COVP).

Theorem 2.6. *An instance of the $(d, 1)$ -AP with cost array C has the COVP if and only if C is a sum-decomposable array with parameters d and 1.*

Proof. Note that one direction follows immediately, i.e. if C is the sum of d vectors, then every feasible solution has the same objective value. Conversely, assume that every feasible solution has the same objective value. For integers $i_1, i_2, \dots, i_d \in \{2, 3, \dots, n\}$ consider the following d pairs of d -tuples:

$$\begin{aligned} & (1, 1, \dots, 1), \quad (i_1, i_2, \dots, i_d) \\ & (i_1, 1, \dots, 1), \quad (1, i_2, \dots, i_d) \\ & (1, i_2, 1, \dots, 1), \quad (i_1, 1, i_3, \dots, i_d) \\ & \vdots \\ & (1, \dots, 1, i_d), \quad (i_1, \dots, i_{d-1}, 1). \end{aligned}$$

There exists a set of $n - 2$ d -tuples which completes each of these pairs to a feasible solution; for example the set $\{(k_1^j, k_2^j, \dots, k_d^j) : j = 2, \dots, n - 1, k_l^j = j \text{ if } j < i_l \text{ and } k_l^j = j + 1 \text{ otherwise, } l = 1, \dots, d\}$. By assumption we have

$$c(i_1, i_2, \dots, i_d) = c(i_1, 1, \dots, 1) + c(1, i_2, \dots, i_d) - c(1, \dots, 1) \quad (7)$$

$$= c(1, i_2, 1, \dots, 1) + c(i_1, 1, i_3, \dots, i_d) - c(1, \dots, 1) \quad (8)$$

$$\begin{aligned} & \vdots \\ & = c(1, \dots, 1, i_d) + c(i_1, \dots, i_{d-1}, 1) - c(1, \dots, 1). \end{aligned}$$

Due to (7) there exist a vector $V_1 = (v_1(i))$ and a $(d - 1)$ -dimensional array $G_1 = (g_1(i_1, \dots, i_{d-1}))$ such that $c(i_1, i_2, \dots, i_d) = v_1(i_1) + g_1(i_2, \dots, i_d)$. Analogously, from (8) it follows that there exists a vector V_2 and a $(d - 1)$ -dimensional array G_2 such that $c(i_1, i_2, \dots, i_d) = v_2(i_2) + g_2(i_1, i_3, \dots, i_d)$. Hence, $c(i_1, i_2, \dots, i_d) = v_1(i_1) + v_2(i_2) + g_{1,2}(i_3, \dots, i_d)$ for some $(d - 2)$ -dimensional array $G_{1,2} = (g_{1,2}(i_1, \dots, i_{d-2}))$. Using the remaining equations in an analogous manner we finally obtain that C is the sum of d vectors, i.e.

$$c(i_1, i_2, \dots, i_d) = v_1(i_1) + v_2(i_2) + \dots + v_d(i_d),$$

where the vectors $V_k = (v_k(i))$ can be chosen as follows:

$$\begin{aligned} v_1(i) &= c(i, 1, \dots, 1) - \frac{d-1}{d}c(1, 1, \dots, 1), \\ & \vdots \\ v_d(i) &= c(1, \dots, 1, i) - \frac{d-1}{d}c(1, 1, \dots, 1). \end{aligned}$$

□

2.4 The COVP for the planar case: $(d, d - 1)$ -AP

We now turn to the planar case. Note that there are exactly two feasible solutions of the $(d, d - 1)$ -AP when $n = 2$.

Definition 2.3. We say that an instance of the $(d, d-1)$ -AP with cost array C has property P_2 if for every $2 \times 2 \times \cdots \times 2$ sub-array of C , which is obtained by restricting the index sets to $\{1, i_1\} \times \{1, i_2\} \times \cdots \times \{1, i_d\}$, the two feasible solutions on the resulting subproblem of size 2 have the same objective value.

Property P_2 and sum-decomposable cost arrays for the $(d, d-1)$ -AP are related in the following way.

Lemma 2.7. *Let I be an instance of the $(d, d-1)$ -AP with cost array C . If I has property P_2 , then C is a sum-decomposable array with parameters d and $d-1$.*

Proof. Consider the $2 \times 2 \times \cdots \times 2$ subarray D_2 of C obtained by restricting index sets to $\{1, i_1\} \times \{1, i_2\} \times \cdots \times \{1, i_d\}$ with $i_j \in \{2, \dots, n\}$ for $j = 1, \dots, d$. By exploiting the fact that the two feasible solutions for the subarray D_2 have the same objective value we get that

$$c(i_1, i_2, \dots, i_d) = \sum_{x \in I_1} c(x) - \sum_{x \in I_2} c(x) + \cdots + (-1)^{d+1} \sum_{x \in I_d} c(x), \quad (9)$$

where I_i is the set of all d -tuples from $\{1, i_1\} \times \{1, i_2\} \times \cdots \times \{1, i_d\}$ with exactly i ones. Then from (9) it follows that C can be expressed as the sum of d $(d-1)$ -dimensional arrays $A_j = (a_j(i_1, \dots, i_{d-1}))$, $j = 1, \dots, d$, defined by

$$\begin{aligned} a_1(i_2, i_3, \dots, i_d) &= \sum_{x \in I_1^1} c(x) - \frac{1}{2} \sum_{x \in I_2^1} c(x) + \cdots + (-1)^{d+1} \frac{1}{d} \sum_{x \in I_d^1} c(x) \\ a_2(i_1, i_3, \dots, i_d) &= \sum_{x \in I_1^2} c(x) - \frac{1}{2} \sum_{x \in I_2^2} c(x) + \cdots + (-1)^{d+1} \frac{1}{d} \sum_{x \in I_d^2} c(x) \\ &\vdots \\ a_d(i_1, i_2, \dots, i_{d-1}) &= \sum_{x \in I_1^d} c(x) - \frac{1}{2} \sum_{x \in I_2^d} c(x) + \cdots + (-1)^{d+1} \frac{1}{d} \sum_{x \in I_d^d} c(x), \end{aligned}$$

where I_i^k is the set of all d -tuples from $\{1, i_1\} \times \{1, i_2\} \times \cdots \times \{1, i_d\}$ with exactly i ones, one of which is on the k -th coordinate. \square

The following result relates property P_2 and the COVP.

Proposition 2.8. *Every instance of the $(d, d-1)$ -AP with cost array C with $n \neq 3$ that has the COVP, also has property P_2 .*

Proof. We will prove that both feasible solutions of the $(d, d-1)$ -AP on the sub-array of C with indices $\{1, 2\} \times \{1, 2\} \times \cdots \times \{1, 2\}$ have the same

objective value; the general case can be shown analogously. When $n = 2$, this is trivially true. Assume $n \geq 4$. We will build two different feasible solutions F_1^d and F_2^d for the $(d, d-1)$ -AP that satisfy the following property: F_1^d and F_2^d both contain a feasible solution of the $(d, d-1)$ -AP on the subproblem induced by the index set $\{1, 2\}^d$, and all other elements of these two solutions are the same. The existence of such F_1^d and F_2^d completes the proof. Namely, by assumption the objective values of F_1^d and F_2^d are equal, hence (9) holds.

Next we explain how F_1^d and F_2^d can be constructed recursively from a feasible solution of the $(d-1, d-2)$ -AP, which we denote by F^{d-1} , which also contains a feasible solution on the subproblem of size 2 induced by the index set $\{1, 2\}^{d-1}$. We define F_j^d , $j = 1, 2$ as follows:

$$F_j^d = \{(i, a_1, a_2, \dots, a_{d-2}, \phi_i^j(a_{d-1})) : (a_1, \dots, a_{d-1}) \in F^{d-1}, i = 1, \dots, n\},$$

where n permutations ϕ_i^j , $i = 1, \dots, n$, are chosen to be mutually disjoint (recall that two permutations α and β are disjoint if $\alpha(i) \neq \beta(i)$ for all i). Furthermore, for every i we choose ϕ_i^1 and ϕ_i^2 such that they coincide except for $\phi_1^1(1) = 1$, $\phi_1^1(2) = 2$, $\phi_2^1(1) = 2$, $\phi_2^1(2) = 1$, in contrast to $\phi_1^2(1) = 2$, $\phi_1^2(2) = 1$, $\phi_2^2(1) = 1$, $\phi_2^2(2) = 2$. To show that such two sets of permutations (for $j = 1$ and $j = 2$) exist, we represent them as two $n \times n$ Latin squares. For $j = 1, 2$, let the j -th table contain the integer $\phi_r^j(s)$ in the row r and column s . The resulting tables will be two Latin squares of order n which are identical except in the 2×2 upper-left corner. That corner is filled with two different Latin squares of order 2, respectively. It is well known that for $n \geq 4$ such Latin squares exist, see [17]. From Observation 2.2 we get that F_j^d are indeed feasible solutions. \square

The approach we followed in the proof of Proposition 2.8 did not serve us to cover the case $n = 3$ and $d \geq 5$. Using a linear algebra approach we were able to cover this case as well and hence to prove the following COVP characterization for the $(d, d-1)$ -AP.

Theorem 2.9. *An instance of the $(d, d-1)$ -AP with cost array C has the COVP if and only if C is a sum-decomposable array with parameters d and $d-1$.*

Proof. If the cost array C is sum-decomposable, then it is straightforward to see that every feasible solution has the same objective value.

Conversely, assume that every feasible solution has the same objective value. For the case $n \neq 3$ the statement follows from Proposition 2.8 and Lemma 2.7. For the remaining case $n = 3$ we make use of the same technique that has been used in [11, 14] to obtain a COVP characterization for the TSP. Let $\mathcal{C}(d, n)$ denote the collection of all d -dimensional $n \times n \times \dots \times n$ cost arrays C for which all feasible solutions of the $(d, d-1)$ -AP have the

same objective value. Clearly $\mathcal{C}(d, n)$ is a linear subspace of the set of all d -dimensional $n \times n \times \cdots \times n$ arrays. Our goal is to prove that

$$\mathcal{C}(d, 3) = \text{SAVS}(d, d-1, 3). \quad (10)$$

To that end, we consider all feasible solutions of the $(d, d-1)$ -AP for $n = 3$. Next we build up the 0-1 matrix M_d where the rows of M_d correspond to the feasible solutions and the columns correspond to the d -tuples over $\{1, 2, 3\}$. The entry of M_d that corresponds to the feasible solution F and the d -tuple (i_1, i_2, \dots, i_d) is set to 1 if and only if $(i_1, i_2, \dots, i_d) \in F$.

Note that every row of the matrix M_{d+1} is obtained from three disjoint rows of the matrix M_d . For every row r_1 of the matrix M_d there are exactly two rows r_2, r_3 disjoint with r_1 , and r_2 and r_3 are also mutually disjoint. Therefore $r_1 r_2 r_3$ and $r_1 r_3 r_2$ are rows of M_{d+1} . Hence the matrix M_{d+1} has twice as many rows as M_d . This corresponds to the fact that for $n = 3$ the number of feasible solutions doubles when moving from the planar d -dimensional assignment problem to the $(d+1)$ -dimensional one. It is easy to see that M_d is a $3 \cdot 2^{d-1} \times 3^d$ matrix. The following matrices M_d , $d = 1, 2$ are provided as illustration:

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array} \right).$$

$\mathcal{C}(d, 3)$ is the solution space of the system of linear equations with coefficient matrix M_d and a constant right hand side vector. Thus we obtain

$$\dim \mathcal{C}(d, 3) = 3^d + 1 - \text{rank}(M_d).$$

From Proposition 2.5 (iv) we know that $\dim(\text{SAVS}(d, d-1, 3)) = 3^d - 2^d$. Hence in order to prove that (10) holds, we need to show that $\text{rank}(M_d) = 2^d + 1$. Observe that in fact it suffices to show that $\text{rank}(M_d) \geq 2^d + 1$ since obviously $\text{SAVS}(d, d-1, 3) \subseteq \mathcal{C}(d, 3)$; Lemma 2.10 below completes the proof. \square

Lemma 2.10. *Let M_d be the matrix constructed above. We have*

$$\text{rank}(M_d) \geq 2^d + 1.$$

Proof. We start with observing the following recursive structure of M_d . Define

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad C_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and recursively for $k \geq 1$

$$A_k = \begin{pmatrix} A_{k-1} & B_{k-1} & C_{k-1} \\ A_{k-1} & C_{k-1} & B_{k-1} \end{pmatrix} \quad B_k = \begin{pmatrix} B_{k-1} & C_{k-1} & A_{k-1} \\ B_{k-1} & A_{k-1} & C_{k-1} \end{pmatrix}$$

$$C_k = \begin{pmatrix} C_{k-1} & A_{k-1} & B_{k-1} \\ C_{k-1} & B_{k-1} & A_{k-1} \end{pmatrix},$$

where A_k , B_k and C_k are $3 \cdot 2^k \times 3^{k+1}$ matrices. It is easy to see that $M_d = A_{d+1}$ for $d \geq 1$. Next we will exhibit a regular $(2^d + 1) \times (2^d + 1)$ submatrix M'_d of M_d which will settle the lemma. We construct new matrices A'_k , B'_k and C'_k from A_k , B_k and C_k as follows: First, remove all columns with indices $\geq 2 \cdot 3^k + 1$. Next, remove all rows and columns with indices that are divisible by 3. It is straightforward to observe that the recursive structure survives this construction. More precisely we have

$$A'_k = \begin{pmatrix} A'_{k-1} & B'_{k-1} \\ A'_{k-1} & C'_{k-1} \end{pmatrix} \quad B'_k = \begin{pmatrix} B'_{k-1} & C'_{k-1} \\ B'_{k-1} & A'_{k-1} \end{pmatrix} \quad C'_k = \begin{pmatrix} C'_{k-1} & A'_{k-1} \\ C'_{k-1} & B'_{k-1} \end{pmatrix} \quad (11)$$

for $k \geq 1$ and

$$A'_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B'_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad C'_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The matrices A'_k , B'_k and C'_k have 2^{k+1} rows and 2^{k+1} columns. We obtain our target matrix M'_d from the matrix A'_{d+1} by re-inserting row 3 and column 3 of the matrix A_{d+1} . In order to show that M'_d is regular, we will calculate its determinant by a recursive approach. We will make use of the observation that the upper left and lower left block are identical in the matrices A'_k , B'_k and C'_k . This will allow us to create a zero block as lower left block of a reduced matrix which has the same determinant as A'_k . This results in

$$\det A'_k = \det A'_{k-1} \det (C'_{k-1} - B'_{k-1}) \quad (12)$$

for $k \geq 1$. An analogous argument yields

$$\det (C'_k - B'_k) = \det (C'_{k-1} - B'_{k-1}) \det (B'_{k-1} + C'_{k-1} - 2A'_{k-1}) \quad (13)$$

and

$$\det (B'_k + C'_k - 2A'_k) = \det (B'_{k-1} + C'_{k-1} - 2A'_{k-1}) \det (3(B'_{k-1} - C'_{k-1})) \quad (14)$$

for $k \geq 1$. Furthermore observe that

$$\det (3(B'_{k-1} - C'_{k-1})) = 3^{2^k} \det (C'_{k-1} - B'_{k-1}) \quad (15)$$

as the involved matrices are of size $2^k \times 2^k$. Let

$$z_k = \det A'_k, \quad u_k = \det (C'_k - B'_k), \quad v_k = \det (B'_k + C'_k - 2A'_k).$$

By explicit calculations we get the initial values $z_0 = 1, u_0 = -1, v_0 = 3$. From (12)–(15) we obtain the following recursions for $k \geq 1$

$$z_k = z_{k-1}u_{k-1}, \quad u_k = u_{k-1}v_{k-1}, \quad v_k = 3^{2^k}v_{k-1}u_{k-1}. \quad (16)$$

By combining the second and the third equation in (16) we obtain $v_k = 3^{2^k}u_k$ which allows to eliminate v_k . We obtain the new system of recursions

$$z_k = z_{k-1}u_{k-1}, \quad u_k = 3^{2^{k-1}}u_{k-1}^2, \quad k \geq 1. \quad (17)$$

This already implies that all matrices A'_k are regular, but for the sake of completeness we provide the solution for the recursion above. It is not hard to show that

$$u_k = 3^{k2^{k-1}}, \quad z_k = 3^{(k-2)2^{k-1}+1}$$

provides a solution to the system (17) with the initial conditions $z_0 = 1$ and $u_0 = -1$. As a consequence thereof we get that

$$\det A'_{d+1} = 3^{d2^{d+1}+1}.$$

Note that M'_d differs from A'_{d+1} only in its additional row and additional column. The additional column (the third column) of M'_d corresponds to the third unit vector. By developing the determinant of M'_d with respect to this column, we obtain

$$\det M'_d = \det A'_{d+1} = 3^{d2^{d+1}+1},$$

which implies that M'_d is regular and hence $\text{rank}(M_d) \geq 2^d + 1$. \square

Let us mention that one can show that $\text{rank}(M_d) = 2^d + 1$ by calculating the reduced row echelon form of matrix M_d . For our purposes it sufficed to show a weaker result which could be obtained more elegantly.

2.5 The COVP for the general case: (d, s) -AP

Theorem 2.6 and Theorem 2.9 impose the following question for the (d, s) -AP.

Question 2.11. *Is it true that a feasible instance of the (d, s) -AP with cost array C has the COVP if and only if C is a sum-decomposable array with parameters d and s ?*

Theorem 2.6 and Theorem 2.9 imply that the answer to Question 2.11 is affirmative in the following cases: $(2, 1)$ -AP, $(3, 1)$ -AP, $(3, 2)$ -AP, $(4, 1)$ -AP and $(4, 3)$ -AP. This leaves us with the $(4, 2)$ -AP as the smallest unsettled case. This is also the smallest case for which it is not guaranteed that a feasible solution exists for all $n \geq 2$.

The following example shows that the answer to Question 2.11 is negative in general.

Example 2.12. There are 72 Graeco-Latin squares of size 3, hence there are 72 feasible solutions for the $(4, 2)$ -AP with $n = 3$, see Observation 2.3. We consider the system of linear equations that is obtained by requiring that all 72 feasible solutions have the same objective value. The dimension of the solution space of this system of equations, and thus the dimension of the space of cost arrays with the COVP, is 49, which can easily be calculated by a computer algebra system. By Proposition 2.5 one gets that the dimension of $\text{SAVS}(4, 2, 3)$ is 33. Hence, there exists a cost array with the COVP that is not sum-decomposable. Now we provide one such array.

Let C be the $3 \times 3 \times 3 \times 3$ array where $c(1, 1, 1, 2)$, $c(1, 1, 2, 1)$, $c(1, 2, 1, 1)$, $c(1, 2, 2, 2)$, $c(2, 1, 1, 1)$, $c(2, 1, 2, 2)$, $c(2, 2, 1, 2)$, $c(2, 2, 2, 1)$ and $c(3, 3, 3, 3)$ have value 1 and all other entries have value 0. All 72 feasible solutions of the $(4, 2)$ -AP with this cost array have the objective value 1, and it is easy to check that C is not sum-decomposable.

We did not find counterexamples for the $(4, 2)$ -AP for $n \geq 4$. For $n = 4$ and $n = 5$ the computer calculations gave the affirmative answer to Question 2.11. For $n = 6$ there are no feasible solutions and for $n = 7$ the number of feasible solutions gets too large to handle.

Conjecture 2.13. *A feasible instance of the $(4, 2)$ -AP with cost array C of size $n \neq 3$ has the COVP if and only if C is a sum-decomposable array with parameters 4 and 2.*

We believe that Example 2.12 occurs since for $n = 3$ the number of feasible solutions is relatively small, but then grows very fast. Note that for larger values of n even the number of Graeco-Latin squares is unknown. This eliminates explicit proof approaches as the set of feasible solutions is not known. A proof would need to exploit the structure of the set of Graeco-Latin squares.

We checked that the answer to Question 2.11 for the $(5, 2)$ -AP for $n = 4$ is affirmative. For $n = 2, 3, 6$ there are no feasible solutions. For $n = 5$ the set of feasible solutions became too large for our straightforward computational approach. The same happened for the $(5, 3)$ -AP for $n = 4$, and for $n = 2, 3$ the problem is again infeasible. The motivation behind our experiments was our wish to obtain a feeling whether the answer to Question 2.11 is affirmative for sufficiently large n . We believe so, but we could handle only very small cases and do not have enough empirical results to propose a conjecture.

3 The COVP for d -dimensional transportation problems

In this section we deal with the COVP for d -dimensional transportation problems. Specifically, we show that our COVP characterization for the axial

d -dimensional assignment problem carries over to the axial d -dimensional transportation problem while this approach fails for the more involved planar case.

Multi-dimensional transportation problems are known in the literature under diverse names. Alternative names are for example multi-index or d -index transportation problems, d -fold transportation problems and multi-way or d -way transportation problems, see e.g. [8, 16].

The d -dimensional transportation problem can be defined along the lines of the definition of the d -dimensional assignment problem (d, s) -AP. We are given a d -dimensional $n_1 \times n_2 \times \dots \times n_d$ cost array C . While in the assignment case the right hand side of all equality constraints is equal to one, in the transportation case we are additionally given an s -dimensional array B^Q for each set $Q \in \mathcal{Q}_s$ of fixed indices which provides the right hand side values for this group of constraints induced by the set Q . The arrays B^Q can be viewed as marginals for the transportation array $X = (x(i_1, i_2, \dots, i_d))$. We refer to the resulting transportation problem as (d, s) -TP.

Like for the assignment case, we obtain the *axial d -dimensional transportation problem* when $s = 1$ and the *planar d -dimensional transportation problem* when $s = d - 1$. As we will deal with the axial d -dimensional transportation problem below, we provide its explicit formulation.

We are given an $n_1 \times n_2 \times \dots \times n_d$ cost array $C = (c(i_1, i_2, \dots, i_d))$ and d supply-demand vectors B_1, \dots, B_d , where the k -th vector $B_k = (b_k(i))$ is an n_k -dimensional vector over the nonnegative integers. Furthermore we assume $\sum_{i=1}^{n_1} b_1(i) = \sum_{i=1}^{n_2} b_2(i) = \dots = \sum_{i=1}^{n_d} b_d(i)$. Let $I_r = \{1, \dots, n_r\}$ be the index set for i_r , $r = 1, \dots, d$. We obtain the following formulation for the $(d, 1)$ -TP:

$$\begin{aligned} \min \quad & \sum_{i_1 \in I_1} \sum_{i_2 \in I_2} \dots \sum_{i_d \in I_d} c(i_1, i_2, \dots, i_d) x(i_1, i_2, \dots, i_d) \\ \text{s.t.} \quad & \sum_{\substack{i_1 \in I_1, \dots, i_d \in I_d \\ \text{s.t. } i_k = j}} x(i_1, i_2, \dots, i_d) = b_k(j) \text{ for all } k \in \{1, \dots, d\}, j \in \{1, \dots, n_k\} \\ & x(i_1, i_2, \dots, i_d) \geq 0 \quad \text{for all } i_r = 1, \dots, n_r, \ r = 1, \dots, d. \end{aligned}$$

If $X = (x(i_1, \dots, i_n))$ has to be integral, the problem above becomes NP-hard for $d \geq 3$. For $d = 2$ the well-known classical Hitchcock transportation problem arises.

Theorem 3.1. *An instance of the axial d -dimensional transportation problem with cost array C has the COVP if and only if C is sum-decomposable array with parameters d and 1.*

Proof. Any instance of the integral axial d -dimensional transportation problem can be transformed into an equivalent instance of the axial d -dimensional assignment problem. To that end, we replace every supply/demand facility

that has a supply/demand value $t > 1$ by t facilities with identical transportation costs that have supply/demand value 1. In this manner we get an equivalent problem with a blown up $n \times n \times \cdots \times n$ cost array where $n = \sum_{i=1}^{n_1} b_1(i)$ and all supplies/demands are 1. Thus the newly obtained problem is the $(d, 1)$ -AP.

For the integral version of the $(d, 1)$ -TP we can apply the COVP characterization from Theorem 2.6 directly. For the non-integral version observe that the transformed problem with unit supplies and demands is a relaxation of the $(d, 1)$ -AP which results if the integrality constraints on X are dropped. In this case it follows from Theorem 2.6 that the set of instances with the COVP is a subspace of $\text{SAVS}(d, 1, n)$, and is hence equal to $\text{SAVS}(d, 1, n)$. Note that the transformation that blows up the cost array and the inverse transformation preserve the sum-decomposability property of the cost array. \square

Note that setting $d = 2$ in Theorem 3.1 implies that an instance of the classical transportation problem with cost matrix C has the COVP if and only if C is a sum matrix. The proof of Theorem 3.1 provides the connection to assignment problems and further to Berenguer's COVP characterization for the TSP, cf. Theorem 1.1. As a by-product this reveals the nature of the connection between results of Klinz and Woeginger [13] on the optimality of the North-West corner rule and Theorem 3.1, and thus answers an open problem mentioned in the concluding section of [13].

At first sight one might expect that Theorem 2.9 for the planar d -dimensional assignment problem $(d, d - 1)$ -AP carries over to the planar d -dimensional transportation problem $(d, d - 1)$ -TP. However several difficulties arise in this case. First, note that the blow-up technique to transform the transportation problem to a (continuous) assignment problem does not work in general in the planar setting. The second and probably bigger obstacle to a COVP characterization for the planar case comes from the fact that for $d \geq 3$ the d -dimensional planar transportation problem does not necessarily have feasible solutions (not even in the non-integral case). Due to the universality result of de Loera and Onn [8] checking feasibility for the 3-dimensional planar (integer) transportation problem is as hard as deciding whether a general linear (integer) program has a feasible solution (the result already holds for a fixed third dimension, i.e., for $n_3 = 3$). As the number of feasible solutions of a feasible instance of the d -dimensional transportation problem can be as small as one, even in the non-integral case, it is not any longer necessary for the COVP that all dual constraints have to be fulfilled with equality. Hence the approach based on the complementarity slackness condition that works for the linear assignment problem and the classical transportation problem, that was explained in the introduction, fails for $d \geq 3$.

Concluding, there does not seem to be much hope to be able to provide

a nice sufficient and necessary condition for the set of instances with the COVP for the 3-dimensional planar transportation problem and even less hope for cases with $d > 3$.

4 The COVP for spanning tree, shortest path and matching problems

In this section we provide COVP characterization for the minimum spanning tree problem, the shortest path problem and the minimum weight maximum cardinality matching problem.

4.1 The COVP for the minimum spanning tree problem

In the minimum spanning tree problem (MST) we are given a connected, undirected graph $G = (V, E)$ and edge weights w_e for each edge $e \in E$. The task is to find a spanning tree for which the sum of edge weights is minimal.

Lemma 4.1. *Let I be an instance of the MST with graph G and weights $w = (w_e)$. If I has the COVP, then every edge in any (simple) cycle in G has the same weight.*

Proof. Let C be a (simple) cycle in G and e be an arbitrary edge from C . There exists a spanning tree T which contains all edges of C except e . By adding edge e to T and removing from T in turn an arbitrary edge $f \neq e$ from C , we obtain another spanning tree T' . As the weights of T and T' are identical, it follows that $w_e = w_f$. Hence all edges in C have the same weight. \square

To formulate the COVP characterization for the MST we need the following definition.

Definition 4.1. Let $G = (V, E)$ be an undirected graph. The undirected graph $H = (V_H, E_H)$ which has a vertex v_e for each edge $e \in E$ and an edge $\{v_e, v_f\} \in E_H$ if and only if e and f lie on a common simple cycle C is called *cycle graph* of G .

Theorem 4.2. *Let I be an instance of the MST problem with graph G and weights $w = (w_e)$. Let H be the cycle graph of G and let V_1, \dots, V_ℓ be the vertex sets of connected components of H and E_1, \dots, E_ℓ be the corresponding sets of edges in G . Then I has the COVP if and only if there exist constants α_i , $i = 1, \dots, \ell$ such that for all $e \in E_i$ $w_e = \alpha_i$, for all i .*

Proof. To prove that the stated condition is sufficient, let T and T' be two spanning trees. It is easy to see that one can move from T to T' by a sequence of moves which add an edge $e \in T' \setminus T$ and delete an edge $f \in T \setminus T'$ where f lies on the unique cycle in $T \cup \{e\}$. As all edges on the cycle have the

same weight, it follows from iterative application that $w(T) = w(T')$, where $w(T)$ denotes the sum of all w_e , $e \in T$.

To prove necessity of the stated condition, first observe that every bridge of G (corresponds to an isolated vertex in H) is part of every spanning tree and hence can have arbitrary weight. The claim now follows by applying Lemma 4.1 for each cycle C in G . \square

Note that if H is connected (which is the case for example if G is 2-connected), then the COVP holds if all edges have the same weight.

It is easy to see that the COVP characterization for the MST problem can be carried over to the setting of matroids (circuits play the role of cycles and bases play the role of spanning trees).

4.2 The COVP for the shortest path problem

Given a weighted graph (undirected or directed) with the vertex set $V = \{1, 2, \dots, n\}$ the shortest path problem is the problem of finding a path from vertex 1 to vertex n such that the sum of edge weights along the path is minimized. In what follows we consider both the undirected and the directed version of the shortest path problem in a complete graph and provide COVP characterizations.

Theorem 4.3. *Let $G = (V, E)$ be the complete undirected graph with the vertex set $V = \{1, 2, \dots, n\}$, $n \geq 3$, and let $w(i, j)$ denote the nonnegative weight of the edge (i, j) . This instance of the undirected shortest path problem has the COVP if and only if the weights are of the following form*

$$w(i, j) = w(j, i) = \begin{cases} a & \text{if } i = 1, j \neq n, \\ b & \text{if } i \neq 1, j = n, \\ a + b & \text{if } i = 1, j = n, \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

for some non-negative reals a and b .

Proof. Assume that every path from 1 to n has the same weight. For $n = 3$ the result is straightforward. Assume $n \geq 4$ and take two distinct vertices i and j such that $1 < i, j < n$. Consider the five paths from vertex 1 to vertex n that only go through a subset of the vertices $\{1, i, j, n\}$. By assumption we get the following relations

$$\begin{aligned} w(1, n) &= w(1, i) + w(i, j) + w(j, n) \\ &= w(1, j) + w(j, i) + w(i, n) \\ &= w(1, i) + w(i, n) \\ &= w(1, j) + w(j, n). \end{aligned}$$

By adding and subtracting appropriate equations we get that $w(i, j) = 0$, $w(1, i) = w(1, j)$, $w(i, n) = w(j, n)$, so (18) follows.

Note that the converse trivially holds, which concludes the proof. \square

Theorem 4.4. *Let $G = (V, E)$ be the complete directed acyclic graph with the vertex set $V = \{1, 2, \dots, n\}$ and edge set $E = \{(i, j) \in V \times V : i < j\}$, and let $w(i, j)$ denote the weight of edge (i, j) . This instance of the directed shortest path problem has the COVP if and only if there exists a real vector $A = (a_i)$ such that*

$$w(i, j) = a_j - a_i \quad \text{for all } i, j \in \{1, \dots, n\}, i < j. \quad (19)$$

Proof. Assume that every path from vertex 1 to vertex n has the same weight. Consider the path composed of edges $(1, i)$, (i, j) and (j, n) . It has the same weight as the path composed of edges $(1, j)$ and (j, n) . It follows that $w(i, j) = w(1, j) - w(1, i)$. Set $a_i := w(1, i)$ for $i = 1, \dots, n$, so that $w(i, j) = a_j - a_i$.

Now assume that for all $i < j$ the weight of (i, j) can be represented as in (19) for some vector $A = (a_i)$. Consider an arbitrary path from vertex 1 to n , and let $1 = v_1 < v_2 < \dots < v_k = n$ be all vertices on that path. Then the weight of the path is

$$\sum_{i=1}^{k-1} w(v_i, v_{i+1}) = \sum_{i=1}^{k-1} a_{v_{i+1}} - a_{v_i} = a_{v_k} - a_{v_1} = a_n - a_1.$$

Since this number is independent of the choice of path, (19) is also sufficient and hence the statement holds. \square

4.3 The COVP for the minimum weight maximum cardinality matching problem

In the minimum weight maximum cardinality matching problem we are given an undirected graph $G = (V, E)$ and edge weights $w(i, j)$ for each edge $(i, j) \in E$. Our goal is to find a matching for which the sum of edge weights is minimal among all matchings of maximal cardinality.

Theorem 4.5. *Let I be an instance of the minimum weight maximum cardinality matching on the complete undirected graph G with n vertices and edge weights $w(i, j)$.*

- (i) *If n is odd, I has the COVP if and only if all edge weights are equal.*
- (ii) *If n is even, I has the COVP if and only if there exists a real vector $A = (a_i)$ such that*

$$w(i, j) = a_i + a_j \quad \text{for all } i \neq j. \quad (20)$$

Proof. Let n be odd. Suppose that every maximum cardinality matching has the same weight. Let $i, j, k \in V$ be three distinct vertices. Let M be a maximum cardinality matching on the vertex set $V \setminus \{i, j, k\}$. By adding an arbitrary edge from the triangle defined by i, j and k to M we obtain a maximum cardinality matching on the initial instance. By assumption it follows that every edge in the triangle defined by i, j and k has the same weight. Hence the statement follows.

Let n be even. Assume that every perfect matching has the same weight. Since each of the two pairs of edges $(i, j), (k, l)$ and $(i, l), (j, k)$ can be identically extended to a perfect matching it follows that

$$w(i, j) + w(k, l) = w(i, l) + w(j, k)$$

for all distinct i, j, k, l . Hence, there exist two real vectors $U = (u_i)$ and $V = (v_i)$ such that $w(i, j) = u_i + v_j$ for all $i \neq j$. Since the weight matrix has to be symmetric (G is undirected), there exists a real vector $A = (a_i)$ such that (20) holds.

Note that (20) is clearly a sufficient condition for the COVP. \square

5 Conclusion

Our goal was to characterize the set of instances with the constant objective value property (COVP), i.e. to investigate the space of all instances for which every feasible solution has the same objective value. The COVP is closely connected to the notion of admissible transformations, a topic studied by various authors.

As our central result, we showed that the COVP instances of the planar and the axial d -dimensional assignment problem are characterized by sum-decomposable arrays with the corresponding parameters. We provided a counterexample which shows that these results do not carry over to general d -dimensional assignment problem. The following remains a challenging open question: Does sum decomposability characterize the COVP instances in all cases of the multidimensional assignment problems for which feasible solutions exist and size of the instance is sufficiently large?

We used the results for the axial d -dimensional assignment problem to characterize the COVP instances for the axial d -dimensional transportation problem.

Furthermore, as simpler side results, we characterized the COVP instances for the following classical combinatorial optimization problems: the minimum spanning tree, the shortest path problem in undirected and directed graphs and the minimum weight cardinality matching problem in complete graphs.

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